

The inversion formula of polylogarithms and the Riemann-Hilbert problem

OI, Shu and UENO, Kimio

Abstract

In this article, we set up a method of reconstructing the polylogarithms $\text{Li}_k(z)$ from zeta values $\zeta(k)$ via the Riemann-Hilbert problem. This is referred to as “a recursive Riemann-Hilbert problem of additive type.” Moreover, we suggest a framework of interpreting the connection problem of the Knizhnik-Zamolodochikov equation of one variable as a Riemann-Hilbert problem.

1 Introduction

Polylogarithms $\text{Li}_k(z)$ ($k \geq 2$) satisfy **the inversion formula**

$$\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \underbrace{\text{Li}_{2,1,\dots,1}}_{k-2}(1-z) = \zeta(k).$$

Applying the Riemann-Hilbert problem of additive type (alternatively, Plemelj-Birkhoff decomposition) [**Bi, Mu, Pl**] to this inversion formula, we show that $\text{Li}_k(z)$ can be reconstructed from boundary values $\zeta(k)$. We prove this by using the Riemann-Hilbert problem recursively so that we refer to this method as **a recursive Riemann-Hilbert problem of additive type**.

As a generalization of this method, we can reconstruct multiple polylogarithms $\text{Li}_{k_1,\dots,k_r}(z)$ from multiple zeta values $\zeta(k_1, \dots, k_r)$. This is nothing but interpreting **the connection relation**[**OiU**]

$$\mathcal{L}(z) = \mathcal{L}^{(1)}(z) \Phi_{KZ}$$

between the fundamental solutions of **the Knizhnik-Zamolodochikov equation of one variable** (KZ equation, for short)

$$\frac{dG}{dz} = \left(\frac{X_0}{z} + \frac{X_1}{1-z} \right) G$$

as a Riemann-Hilbert problem. Here Φ_{KZ} is **Drinfel'd associator** and $\mathcal{L}(z)$ (resp. $\mathcal{L}^{(1)}(z)$) is the fundamental solution of KZ equation normalized at $z=0$ (resp. $z=1$). We have completely solved this problem and a preprint is now in preparation.

2010 *Mathematics Subject Classification*. Primary 34M50, 11G55; Secondary 30E25, 11M06, 32G34;

Acknowledgment The first author is supported by Waseda University Grant for Special Research Projects No. 2011B-095. The second author is partially supported by JSPS Grant-in-Aid No. 22540035.

2 The inversion formula of polylogarithms

For positive integers k , polylogarithms $\text{Li}_k(z)$ are introduced as follows: First we set $\text{Li}_1(z) = -\log(1-z)$. In the domain $D = \mathbf{C} \setminus \{z = x \mid 1 \leq x\}$, $\text{Li}_1(z)$ has a branch such that $\text{Li}_1(0) = 1$ (the principal value of $\text{Li}_1(z)$). Starting from the principal value of $\text{Li}_1(z)$, we introduce $\text{Li}_k(z)$, which are holomorphic on D , recursively by

$$\text{Li}_k(z) = \int_0^z \frac{\text{Li}_{k-1}(t)}{t} dt \quad (k \geq 2). \quad (1)$$

where the integral contour is assumed to be in D . Then $\text{Li}_k(z)$ has a Taylor expansion

$$\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \quad (2)$$

on $|z| < 1$. We obtain, for $k \geq 2$,

$$\lim_{z \rightarrow 1, z \in D} \text{Li}_k(z) = \zeta(k), \quad (3)$$

where $\zeta(k)$ is the Riemann zeta value $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

From (1), we have differential recursive relations:

$$\frac{d}{dz} \text{Li}_1(z) = \frac{1}{1-z}, \quad \frac{d}{dz} \text{Li}_k(z) = \frac{\text{Li}_{k-1}(z)}{z} \quad (k \geq 2). \quad (4)$$

By virtue of (1), $\text{Li}_k(z)$ is analytically continued to a many-valued analytic function on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. However, in this article, we will use the notation $\text{Li}_k(z)$ as the principal value stated previously.

We also define multiple polylogarithms $\text{Li}_{2,1,\dots,1}(z)$ ($k \geq 2$) as

$$\text{Li}_{\underbrace{2,1,\dots,1}_{k-2}}(z) = \int_0^z \frac{(-1)^{k-1} \log^{k-1}(1-t)}{(k-1)! t} dt. \quad (5)$$

By using these relations and (3), one can obtain easily **the inversion formula** of polylogarithms.

Proposition 1 (the inversion formula of polylogarithms). *For $k \geq 2$, the following functional relation holds.*

$$\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \text{Li}_{\underbrace{2,1,\dots,1}_{k-2}}(1-z) = \zeta(k). \quad (6)$$

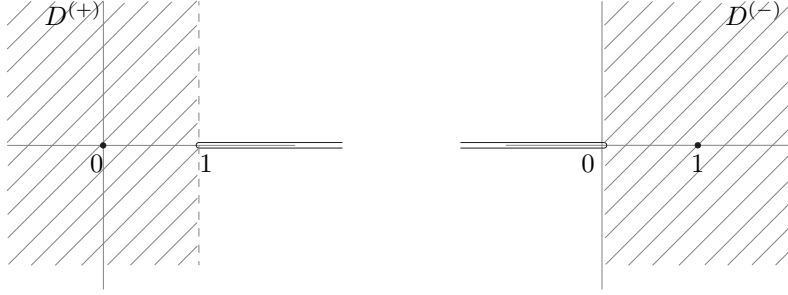


Figure 1: The domains $D^{(+)}$, $D^{(-)}$.

Proof. Differentiating the left hand side of the equation (6), we have

$$\frac{d}{dz} \left(\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \underbrace{\text{Li}_{2,1,\dots,1}}_{k-2}(1-z) \right) = 0.$$

Therefore the left hand side of (6) is a constant. Taking the limit of the left hand side of (6) as $z \in D$ tends to 1 and using (3), we see that the constant is equal to $\zeta(k)$. \square

The branch of $\text{Li}_{2,\underbrace{1,\dots,1}_{k-2}}(1-z)$ on the domain $D' = \mathbf{C} \setminus \{z = x \mid x \leq 0\}$ is determined from the principal value of $\log z$.

3 The recursive Riemann-Hilbert problem of additive type

Let $D^{(+)}$, $D^{(-)}$ be domains of \mathbf{C} defined by

$$D^{(+)} = \{z = x + yi \mid x < 1, -\infty < y < \infty\} \subset D,$$

$$D^{(-)} = \{z = x + yi \mid 0 < x, -\infty < y < \infty\} \subset D'.$$

The following theorem says that polylogarithms $\text{Li}_k(z)$ are characterized by the inversion formula.

Theorem 2. Put $f_1^{(+)}(z) = \text{Li}_1(z)$. For $k \geq 2$, we assume that $f_k^{(\pm)}(z)$ are holomorphic functions on $D^{(\pm)}$ satisfying the functional relation

$$f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} f_{k-j}^{(+)}(z) + f_k^{(-)}(z) = \zeta(k) \quad (z \in D^{(+)} \cap D^{(-)}), \quad (7)$$

the asymptotic conditions

$$\frac{d}{dz} f_k^{(\pm)}(z) \rightarrow 0 \quad (z \rightarrow \infty, z \in D^{(\pm)}), \quad (8)$$

and the normalization condition

$$f_k^{(+)}(0) = 0. \quad (9)$$

Then we have

$$f_k^{(+)}(z) = \text{Li}_k(z), \quad f_k^{(-)}(z) = \text{Li}_{\underbrace{2,1,\dots,1}_{k-2}}(1-z) \quad (k \geq 2).$$

Proof. We prove the theorem by induction on $k \geq 2$. For the case $k = 2$, the proof can be done in the same manner as the case $k > 2$ from the definition of $f_1^{(+)}(z)$. So we assume that $f_j^{(+)}(z) = \text{Li}_j(z)$ and $f_j^{(-)}(z) = \text{Li}_{\underbrace{2,1,\dots,1}_{j-2}}(1-z)$ for

$2 \leq j \leq k-1$. Now we show that $f_k^{(+)}(z) = \text{Li}_k(z)$, $f_k^{(-)}(z) = \text{Li}_{\underbrace{2,1,\dots,1}_{k-2}}(1-z)$.

From the assumption, the equation (7) becomes

$$f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + f_k^{(-)}(z) = \zeta(k). \quad (10)$$

Differentiating this equation, we have

$$\begin{aligned} 0 &= \frac{d}{dz} \left(f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + f_k^{(-)}(z) \right) \\ &= \frac{d}{dz} f_k^{(+)}(z) + \sum_{j=1}^{k-2} \left(\frac{1}{z} \frac{(-1)^j \log^{j-1} z}{(j-1)!} \text{Li}_{k-j}(z) + \frac{(-1)^j \log^j z}{j!} \frac{\text{Li}_{k-j-1}(z)}{z} \right) \\ &\quad + \frac{1}{z} \frac{(-1)^{k-1} \log^{k-2} z}{(k-2)!} \text{Li}_1(z) + \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} \frac{1}{1-z} \\ &\quad + \frac{d}{dz} f_k^{(-)}(z) \\ &= \frac{d}{dz} f_k^{(+)}(z) - \frac{\text{Li}_{k-1}(z)}{z} + \frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} + \frac{d}{dz} f_k^{(-)}(z). \end{aligned}$$

Thus we obtain

$$\frac{d}{dz} f_k^{(+)}(z) - \frac{\text{Li}_{k-1}(z)}{z} = -\frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} - \frac{d}{dz} f_k^{(-)}(z) \quad (11)$$

on $z \in D^{(+)} \cap D^{(-)}$. Here, the left hand side of (11) is holomorphic on $D^{(+)}$ and the right hand side of (11) is holomorphic on $D^{(-)}$. Therefore the both sides of (11) are entire functions. Using the asymptotic condition (8) and

$$\frac{\text{Li}_{k-1}(z)}{z} \rightarrow 0 \quad (z \rightarrow \infty, z \in D^{(+)}), \quad \frac{\log^{k-1} z}{1-z} \rightarrow 0 \quad (z \rightarrow \infty, z \in D^{(-)}),$$

we have that both sides of (11) are 0 by virtue of Liouville's theorem. Therefore we have

$$\begin{aligned} f_k^{(+)}(z) &= \int^z \frac{\text{Li}_{k-1}(z)}{z} dz = \text{Li}_k(z) + c_k^{(+)}, \\ f_k^{(-)}(z) &= \int^z -\frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} dz = \text{Li}_{\underbrace{2,1,\dots,1}_{k-2}}(1-z) + c_k^{(-)}, \end{aligned}$$

where $c_k^{(+)}, c_k^{(-)}$ are integral constants. From the normalization condition (9), it is clear that $c_k^{(+)}$ is equal to 0. Finally, substituting $f_k^{(+)}(z)$ and $f_k^{(-)}(z)$ in (7), we obtain

$$\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \underbrace{\text{Li}_{2,1,\dots,1}}_{k-2}(1-z) + c_k^{(-)} = \zeta(k). \quad (12)$$

Comparing the inversion formula (6), we have $c_k^{(-)} = 0$. This concludes the proof. \square

The equation (10) is interpreted as the decomposition of the holomorphic function

$$\sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z)$$

on $z \in D^{(+)} \cap D^{(-)}$ to a sum of a function $f_k^{(+)}(z)$, which is holomorphic on $D^{(+)}$, and a function $f_k^{(-)}(z)$, which is holomorphic on $D^{(-)}$. This decomposition is nothing but a Riemann-Hilbert problem of additive type. The theorem says that polylogarithms $\text{Li}_k(z)$ can be constructed from the boundary value $\zeta(k)$ by applying this Riemann-Hilbert problem recursively. In this sense, we call (7) **the recursive Riemann-Hilbert problem of additive type**.

References

- [Bi] G. D. Birkhoff, The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, Proc. Am. Acad. Arts and Sciences, **49** (1914), 521-568.
- [Mu] N. I. Muskhelishvili, Singular Integral Equations, P. Noordhoff Ltd. (1946).
- [OiU] S. Oi and K. Ueno, Connection Problem of Knizhnik-Zamolodchikov Equation on Moduli Space $\mathcal{M}_{0,5}$, preprint (2011) arXiv:1109.0715.
- [Pl] J. Plemelj, Problems in the sense of Riemann and Klein, Interscience Tracts in Pure and Applied Mathematics, No. 16, Interscience Publishers, John Wiley & Sons Inc. New York-London-Sydney (1964).

OI, Shu.

Department of Mathematics, School of Fundamental Sciences and Engineering,
Faculty of Science and Engineering, Waseda university. 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan.

e-mail: shu_oi@toki.waseda.jp

UENO, Kimio

Department of Mathematics, School of Fundamental Sciences and Engineering,
Faculty of Science and Engineering, Waseda university. 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan.

e-mail: uenoki@waseda.jp